Directional Derivatives in the Plane

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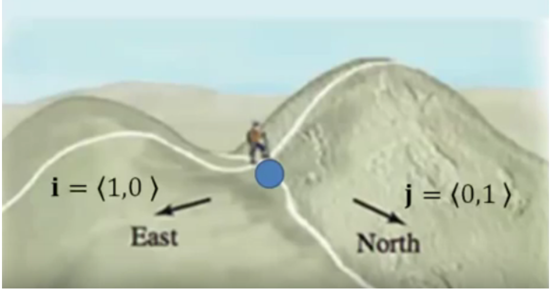
Overview

If we look at the map showing contours along the Hudson River in New York, we will notice that the tributary streams flow perpendicular to the contours. The streams are following paths of steepest descent so the waters reach the Hudson as quickly as possible. Therefore, the fastest instantaneous rate of change in a stream's elevation above sea level has a particular direction. In this lecture, we will see why this direction, called the *downhill direction*, is perpendicular to the contours.

The figure shows contours along the Hudson River in New York show streams, which follow paths of steepest descent, running perpendicular to the contours.

Let $z = f(x, y)$ be the surface shown and assume the hiker is at the point

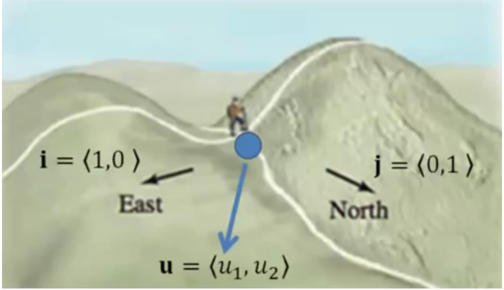
$$(a, b, f(a, b)).$$



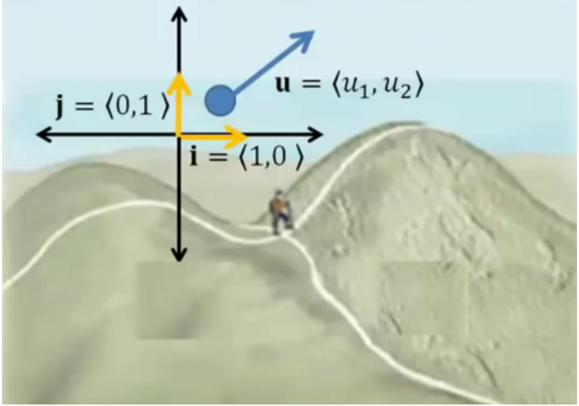
Directional Derivatives in the Plane

The hiker would like to travel on the surface in the direction of unit vector

$$\mathsf{u} = u_1\mathsf{i} + u_2\mathsf{j}.$$



Let s be the horizontal distance traveled.



The change in elevation with respect to horizontal distance traveled is

$$\frac{d}{ds}f(a+su_1, b+su_2).$$

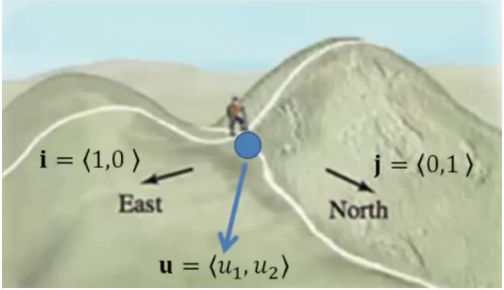
By chain rule, we get

$$\begin{aligned}
\left(\frac{df}{ds}\right)_{\boldsymbol{u},(a,b)} &= \left(\frac{\partial f}{\partial x}\right)_{(a,b)}\frac{dx}{ds} + \left(\frac{\partial f}{\partial y}\right)_{(a,b)}\frac{dy}{ds} \\
&= \left(\frac{\partial f}{\partial x}\right)_{(a,b)}.u_1 + \left(\frac{\partial f}{\partial y}\right)_{(a,b)}.u_2 \\
&= \left[\left(\frac{\partial f}{\partial x}\right)_{(a,b)}\boldsymbol{i} + \left(\frac{\partial f}{\partial y}\right)_{(a,b)}\boldsymbol{j}\right].\left[u_1\boldsymbol{i} + u_2\boldsymbol{j}\right]
\end{aligned}$$

Directional Derivatives in the Plane

$$\left(\frac{df}{ds}\right)_{\boldsymbol{u},(a,b)} = \left[\left(\frac{\partial f}{\partial x}\right)_{(a,b)}\boldsymbol{i} + \left(\frac{\partial f}{\partial y}\right)_{(a,b)}\boldsymbol{j}\right]\cdot\left[u_1\boldsymbol{i} + u_2\boldsymbol{j}\right]$$

gives the rate of change of elevation as the hiker travels in the direction of u at the moment the hiker is at location $(a, b, f(a, b))$.



Recall that if $z = f(x, y)$, then the partial derivatives f_x and f_y are defined as

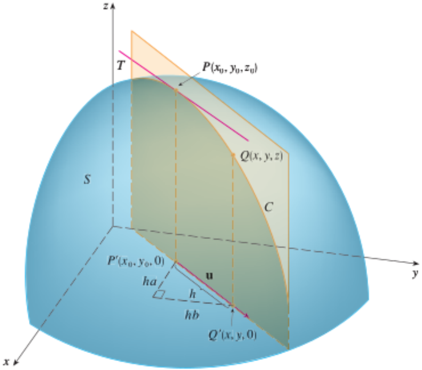
$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$
$$f_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

and represent the rates of change of z in the x- and y-directions, that is, in the directions of the unit vectors i and j.

Suppose that we now wish to find the rate of change of z at (x_0, y_0) in the direction of an arbitrary unit vector $u = u_1\mathrm{i} + u_2\mathrm{j}$. To do this we consider the surface S with equation $z = f(x, y)$ (the graph of f) and we let $z_0 = f(x_0, y_0)$.

Then the point $P(x_0, y_0, z_0)$ lies on S. The vertical plane that passes through P in the direction of u intersects S in a curve C.

The slope of the tangent line T to C at the point P is the rate of change of z in the direction of u.



Directional Derivatives in the Plane

We know that if $f(x, y)$ is differentiable, then the rate at which f changes with respect to t along a differentiable curve $x = g(t), y = h(t)$ is

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

At any point

$$P_0(x_0, y_0) = P_0(g(t_0), h(t_0)),$$

this equation gives the rate of change of f with respect to increasing t and therefore depends, among other things, on the direction of motion along the curve.

If the curve is a straight line and t is the arc length parameter along the line measured from P_0 in the direction of a given unit vector $\boldsymbol{u}$, then

$$df/dt$$

is the rate of change of f with respect to distance in its domain in the direction of $\boldsymbol{u}$.

By varying $\boldsymbol{u}$, we find the rates at which f changes with respect to distance as we move through P_0 in different directions.

Suppose that the function $f(x, y)$ is defined throughout a region R in the xy-plane, that $P_0(x_0, y_0)$ is a point in R, and that

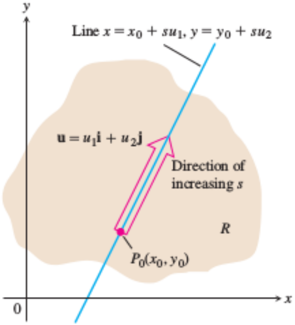
$$\boldsymbol{u} = u_1\boldsymbol{i} + u_2\boldsymbol{j}$$

is a unit vector.

Then the equations

$$x = x_0 + su_1, \quad y = y_0 + su_2$$

parameterize the line through P_0 parallel to $\boldsymbol{u}$. If the parameter s measures arc length from P_0 in the direction of $\boldsymbol{u}$, we find the rate of change of f at P_0 in the direction of $\boldsymbol{u}$ by calculating df/ds at P_0.



Definition 1.

The derivative of f at $P_0(x_0, y_0)$ in the direction of the unit vector $u = u_1\boldsymbol{i} + u_2\boldsymbol{j}$ *is the number*

$$\left(\frac{df}{ds}\right)_{\boldsymbol{u},P_0} = \lim_{s\to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s} \tag{1}$$

provided the limit exists.

The directional derivative is also denoted by $(D_{\boldsymbol{u}} f)_{P_0}$.

The partial derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ are the directional derivatives of f at P_0 in the **i** and **j** directions. This observation can be seen by comparing Equation (1) to the definitions of the two partial derivatives.

Example 2.

Using the definition, find the derivative of

$$f(x, y) = x^2 + xy$$

at $P_0(1, 2)$ *in the direction of the unit vector* $\mathrm{u} = (1/\sqrt{2})\mathrm{i} + (1/\sqrt{2})\mathrm{j}$.

Solution

Applying the definition in Equation (1), we obtain

$$\begin{aligned}
\left(\frac{df}{ds}\right)_{\mathbf{u},P_0} &= \lim_{s\to 0} \frac{f(x_0+su_1, y_0+su_2) - f(x_0,y_0)}{s} \\
&= \lim_{s\to 0} \frac{\left(1+\frac{s}{\sqrt{2}}\right)^2 + \left(1+\frac{s}{\sqrt{2}}\right)\left(2+\frac{s}{\sqrt{2}}\right) - (1^2 + 1\cdot 2)}{s} \\
&= \lim_{s\to 0} \frac{\left(1+\frac{2s}{\sqrt{2}}+\frac{s^2}{2}\right) + \left(2+\frac{3s}{\sqrt{2}}+\frac{s^2}{2}\right) - 3}{s} \\
&= \lim_{s\to 0} \frac{\frac{5s}{\sqrt{2}} + s^2}{s} = \lim_{s\to 0}\left(\frac{5}{\sqrt{2}} + s\right) = \frac{5}{\sqrt{2}}.
\end{aligned}$$

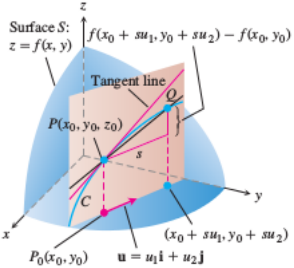
the rate of change of $f(x,y) = x^2 + xy$ at $P_0(1,2)$ in the direction $\mathbf{u}$ is $5/\sqrt{2}$.

The equation $z = f(x, y)$ represents a surface S in space. If

$$z_0 = f(x_0, y_0),$$

then the point $P_0(x_0, y_0, z_0)$ lies on S.

The vertical plane that passes through P and $P_0(x_0, y_0)$ parallel to $\boldsymbol{u}$ intersects S in a curve C. The rate of change of f in the direction of $\boldsymbol{u}$ is the slope of the tangent to C at P in the right-handed system formed by the vectors $\boldsymbol{u}$ and $\boldsymbol{k}$.



Interpretation of the Directional Derivative

When $\boldsymbol{u} = \boldsymbol{i}$, the directional derivative at P_0 is $\partial f/\partial x$ evaluated at (x_0, y_0). When $\boldsymbol{u} = \boldsymbol{j}$, the directional derivative at P_0 is $\partial f/\partial y$ evaluated at (x_0, y_0).

The directional derivative generalizes the two partial derivatives. We can now ask for the rate of change of f in any direction $\boldsymbol{u}$, not just the directions $\boldsymbol{i}$ and $\boldsymbol{j}$.

Here's a physical interpretation of the directional derivative. Suppose that

$$T = f(x, y)$$

is the temperature at each point (x, y) over a region in the plane.

Then $f(x_0, y_0)$ is the temperature at the point $P_0(x_0, y_0)$ and $(D_{\boldsymbol{u}} f)_{P_0}$ is the instantaneous rate of change of the temperature at P_0 stepping off in the direction $\boldsymbol{u}$.

Calculation and Gradients

We now develop an efficient formula to calculate the directional derivative for a differentiable function f. We begin with the line

$$x = x_0 + su_1, \quad y = y_0 + su_2$$

through $P_0(x_0, y_0)$, parameterized with the arc length parameter s increasing in the direction of the unit vector $\boldsymbol{u} = u_1\boldsymbol{i} + u_2\boldsymbol{j}$. Then

$$\begin{aligned}
\left(\frac{df}{ds}\right)_{\boldsymbol{u},P_0} &= \left(\frac{\partial f}{\partial x}\right)_{P_0}\frac{dx}{ds} + \left(\frac{\partial f}{\partial y}\right)_{P_0}\frac{dy}{ds} \\
&= \left(\frac{\partial f}{\partial x}\right)_{P_0}.u_1 + \left(\frac{\partial f}{\partial y}\right)_{P_0}.u_2 \\
&= \left[\left(\frac{\partial f}{\partial x}\right)_{P_0}\boldsymbol{i} + \left(\frac{\partial f}{\partial y}\right)_{P_0}\boldsymbol{j}\right].\left[u_1\boldsymbol{i} + u_2\boldsymbol{j}\right]
\end{aligned}$$

The above equation says that the derivative of a differentiable function f in the direction of **u** at P_0 is the dot product of **u** with the special vector called the *gradient* of f at P_0.

Definition 3.

The gradient vector (gradient) *of* $f(x, y)$ *at a point* $P_0(x_0, y_0)$ *is the vector*

$$\nabla f = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j}$$

obtained by evaluating the partial derivatives of f *at* P_0*.*

The notation ∇f is read "grad f" as well as "gradient of f" and "del f." The symbol ∇ itself is read "del." Another notation for the gradient is grad f, read the say it is written.

The Directional Derivative Is a Dot Product

Theorem 4.

If $f(x, y)$ is differentiable in an open region containing $P_0(x_0, y_0)$, then the derivative of f in the direction of $\boldsymbol{u}$ at P_0 is the dot product of $\boldsymbol{u}$ with the gradient of f at P_0:

$$\left(\frac{df}{ds}\right)_{\boldsymbol{u},P_0} = (\nabla f)_{P_0}.\boldsymbol{u}.$$

Example 5.

Find the derivative of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$ in the direction of $\mathrm{v} = 3\mathrm{i} - 4\mathrm{j}$.

Solution

The direction of $\mathbf{v}$ is the unit vector obtained by dividing $\mathbf{v}$ by its length:

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{v}}{5} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

The partial derivatives of f are everywhere continuous and at $(2,0)$ are given by

$$f_x(2,0) = (e^y - y\sin(xy))_{(2,0)} = e^0 - 0 = 1$$
$$f_y(2,0) = (xe^y - x\sin(xy))_{(2,0)} = 2e^0 - 2\cdot 0 = 2.$$

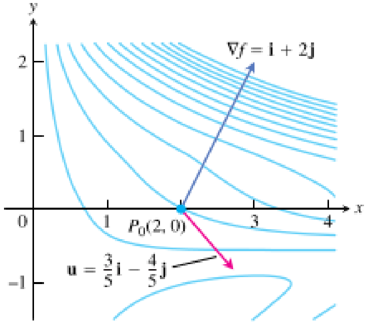
The gradient of f at $(2,0)$ is

$$\nabla f|_{(2,0)} = f_x(2,0)\mathbf{i} + f_y(2,0)\mathbf{j} = \mathbf{i} + 2\mathbf{j}.$$

Solution (contd...)

The derivative of f at $(2, 0)$ in the direction of $\mathbf{v}$ is therefore

$$\begin{aligned}(D_{\mathbf{u}} f)|_{(2,0)} &= \nabla f|_{(2,0)} \cdot \mathbf{u} \\ &= (\mathbf{i} + 2\mathbf{j}) \cdot \left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right) = \frac{3}{5} - \frac{8}{5} = -1.\end{aligned}$$



The Directional Derivative Is a Dot Product

Evaluating the dot product in the formula

$$D_{\mathbf{u}} f = (\nabla f).\boldsymbol{u} = |\nabla f| \; |\boldsymbol{u}| \; \cos\theta = |\nabla f| \; \cos\theta,$$

where θ is the angle between the vectors $\boldsymbol{u}$ and ∇f, reveals the following properties.

1. The function f increases most rapidly when $\cos\theta = 1$ or when $\boldsymbol{u}$ is the direction of ∇f. That is, at each point P in its domain, f increases most rapidly in the direction of the gradient ∇f at P. The derivative in this direction is

$$D_{\boldsymbol{u}} f = |\nabla f| \ \cos(0) = |\nabla f|.$$

2. Similarly, f decreases most rapidly in the direction of $-\nabla f$. The derivative in this direction is

$$D_{\boldsymbol{u}} f = |\nabla f| \ \cos(\pi) = -|\nabla f|.$$

3. Any direction $\boldsymbol{u}$ orthogonal to a gradient $\nabla f \neq 0$ is a direction of zero change in f because θ then equals $\pi/2$ and

$$D_{\boldsymbol{u}} f = |\nabla f| \ \cos(\pi/2) = |\nabla f|.0 = 0.$$

The above properties hold in three dimensions as well as two.

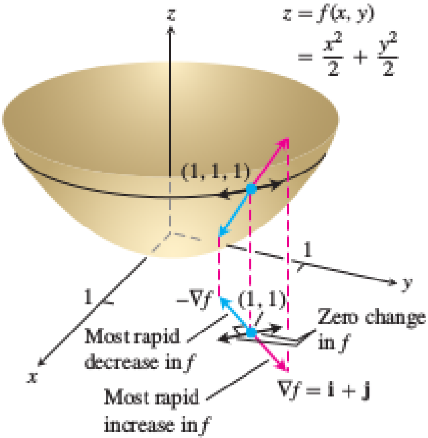
Example 6.

Find the directions in which $f(x, y) = \frac{x^2}{2} + \frac{y^2}{2}$.

(a) *increases most rapidly at the point* $(1, 1)$.

(b) *decreases most rapidly at* $(1, 1)$.

(c) *What are the directions of zero change in* f *at* $(1, 1)$*?*


z
z = f(x, y)
= x^2/2 + y^2/2
(1, 1, 1)
1
y
1
x
−∇f
(1, 1)
Zero change
in f
Most rapid
decrease in f
Most rapid
increase in f
∇f = i + j

Solution

(a) The function increases most rapidly in the direction of ∇f at (1,1). The gradient there is

$$(\nabla f)_{(1,1)} = (x\mathbf{i} + y\mathbf{j})_{(1,1)} = \mathbf{i} + \mathbf{j}.$$

Its direction is

$$\mathbf{u} = \frac{\mathbf{i} + \mathbf{j}}{|\mathbf{i} + \mathbf{j}|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{(1)^2 + (1)^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}.$$

(b) The function decreases most rapidly in the direction of $-\nabla f$ at (1,1), which is

$$-\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

(c) The directions of zero change at (1,1) are the directions orthogonal to ∇f:

$$\mathbf{n} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \quad \text{and} \quad -\mathbf{n} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

Example 7.

The directional derivative of

$$f(x, y) = \frac{x^2 + y^2}{4}$$

at $(3, 2)$ *in the direction of*

$$\mathrm{u} = \frac{1}{\sqrt{2}}\mathrm{i} + \frac{1}{\sqrt{2}}\mathrm{j}$$

is about 2.47 *(Verify !)*

Moving in the direction of $\mathrm{u} = \frac{1}{\sqrt{2}}\mathrm{i} + \frac{1}{\sqrt{2}}\mathrm{j}$ *the rate of change is about* 2.5 *units* UP *for every unit along* u.

Directional Derivative - An Example

Example 8.

The directional derivative of

$$f(x,y) = \frac{x^2 + y^2}{4}$$

at $(3, 2)$ *in the direction of*

$$\mathrm{u} = \frac{1}{2}\mathrm{i} - \frac{\sqrt{3}}{2}\mathrm{j}$$

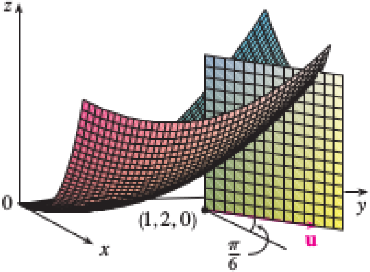
is about -0.98 *(Verify !)*

Moving in the direction of $\mathrm{u} = \frac{1}{2}\mathrm{i} - \frac{\sqrt{3}}{2}\mathrm{j}$ *the rate of change is about* 1 *unit* DOWN *for every unit along* u.

Example 9.

The directional derivative of $f(x,y) = x^3 - 3xy + 4y^2$ *in the direction of the unit vector* u *given by angle* $\theta = \pi/6$ *is*

$$\frac{13 - 3\sqrt{3}}{2}.$$



Moving in the direction of u, the rate of change is about $\frac{13-3\sqrt{3}}{2}$ units **UP** for every unit along u.

Gradients and Tangents to Level Curves

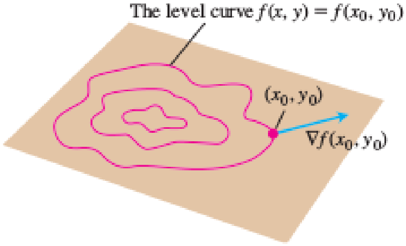
If a differentiable function $f(x, y)$ has a constant value c along a smooth curve $\boldsymbol{r} = g(t)\boldsymbol{i} + h(t)\boldsymbol{j}$ (making the curve a level curve of f), then $f(g(t), h(t)) = c$. Differentiating both sides of this equation with respect to t leads to the equation

$$\begin{aligned}
\frac{d}{dt} f(g(t), h(t)) &= \frac{d}{dt}(c) \\
\frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} &= 0 \\
\Big(\frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j}\Big).\Big(\frac{dg}{dt}\boldsymbol{i} + \frac{dh}{dt}\boldsymbol{j}\Big) &= 0 \\
\nabla f . \frac{d\boldsymbol{r}}{dt} &= 0. \qquad (2)
\end{aligned}$$

Equation (2) says that ∇f is normal to the tangent vector $\frac{d\boldsymbol{r}}{dt}$, so it is normal to the curve.

Gradients and Tangents to Level Curves

At every point (x_0, y_0) in the domain of a differentiable function $f(x, y)$, the gradient of f is normal to the level curve through (x_0, y_0).



Gradients and Tangents to Level Curves

Equation (2) validates our observation that streams flow perpendicular to the contours in topographical maps.

Since the downflowing stream will reach its destination in the fastest way, it must flow in the direction of the negative gradient vectors from Property 2 for the directional derivative. Equation (2) tells us these directions are perpendicular to the level curves.

Tangent Lines to Level Curves

The observation also enables us to find equations for tangent lines to level curves. Tangent lines to level curves are the lines normal to the gradients.

The line through a point $P_0(x_0, y_0)$ normal to a vector

$$\boldsymbol{N} = A\boldsymbol{i} + B\boldsymbol{j}$$

has the equation

$$A(x - x_0) + B(y - y_0) = 0.$$

If $\boldsymbol{N}$ is the gradient

$$(\nabla f)_{(x_0,y_0)} = f_x(x_0, y_0)\boldsymbol{i} + f_y(x_0, y_0)\boldsymbol{j},$$

the equation is the tangent line given by

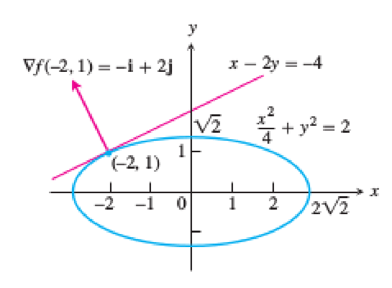
$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0.$$

Tangent Lines to Level Curves

We can find the tangent to the ellipse $(x^2/4) + y^2 = 2$ by treating the ellipse as a level curve of the function $f(x, y) = (x^2/4) + y^2$.

Example 10.

Find an equation for the tangent to the ellipse $\frac{x^2}{4} + y^2 = 2$ *at the point* $(-2, 1)$.



Solution

The ellipse is a level curve of the function

$$f(x,y) = \frac{x^2}{4} + y^2.$$

The gradient of f at (-2,1) is

$$\nabla f|_{(-2,1)} = \left(\frac{x}{2}\mathbf{i} + 2y\mathbf{j}\right)_{(-2,1)} = -\mathbf{i} + 2\mathbf{j}.$$

The tangent is the line

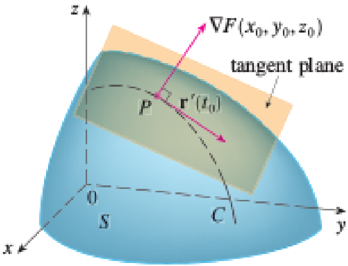
$$\begin{aligned}(-1)(x+2) + (2)(y-1) &= 0\\ x - 2y &= -4.\end{aligned}$$

Tangent Planes to Level Surfaces

If $\nabla F(x_0, y_0, z_0) \neq 0$, it is therefore natural to define the **tangent plane to the level surface** $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$ as the plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$.

Using the standard equation of a plane, we can write the equation of this tangent plane as

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0.$$



Algebra Rules for Gradients

If we know the gradients of two functions f and g, we automatically know the gradients of their constant multiples, sum, difference, product and quotient.

Notice that these rules have the same form as the corresponding rules for derivatives of single-variable functions.

1. Constant Multiple Rule : $\nabla(kf) = k\nabla f \quad$ (any number k)
2. Sum Rule : $\nabla(f + g) = \nabla f + \nabla g$
3. Difference Rule : $\nabla(f - g) = \nabla f - \nabla g$
4. Product Rule : $\nabla(fg) = f\nabla g + g\nabla f$
5. Quotient Rule : $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$.

Example 11.

We illustrate two of the rules with

$$f(x,y) = x - y \qquad g(x,y) = 3y$$
$$\nabla f = \mathrm{i} - \mathrm{j} \qquad \nabla g = 3\mathrm{j}.$$

We have

1. $\nabla(f-g) = \nabla(x-4y) = \mathbf{i} - 4\mathbf{j} = \nabla f - \nabla g$ Rule 2
2. $$\begin{aligned} \nabla(fg) &= \nabla(3xy - 3y^2) = 3y\mathbf{i} + (3x-6y)\mathbf{j} \\ &= 3y(\mathbf{i}-\mathbf{j}) + 3y\mathbf{j} + (3x-6y)\mathbf{j} \\ &= 3y(\mathbf{i}-\mathbf{j}) + (3x-3y)\mathbf{j} \\ &= 3y(\mathbf{i}-\mathbf{j}) + (x-y)3\mathbf{j} = g\nabla f + f\nabla g. \end{aligned}$$

Functions of Three Variables

For a differentiable function $f(x, y, z)$ and a unit vector $\boldsymbol{u} = u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k}$ in space, we have

$$\nabla f = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}$$

and

$$D_{\boldsymbol{u}}f = (\nabla f).\boldsymbol{u} = \frac{\partial f}{\partial x}u_1 + \frac{\partial f}{\partial y}u_2 + \frac{\partial f}{\partial z}u_3.$$

The directional derivative can once again be written in the form

$$D_{\boldsymbol{u}}f = \nabla f.\boldsymbol{u} = |\nabla f| \ \cos\theta,$$

so the properties listed earlier for functions of two variables continue to hold. At any given point, f increases most rapidly in the direction of ∇f and decreases most rapidly in the direction of $-\nabla f$. In any direction orthogonal to ∇f, the derivative is zero.

Example 12.

(a) *Find the derivative of* $f(x, y, z) = x^3 - xy^2 - z$ *at* $P_0(1, 1, 0)$ *in the direction of* $\mathrm{v} = 2\mathrm{i} - 3\mathrm{j} + 6\mathrm{k}$.

(b) *In what directions does* f *change most rapidly at* P_0*, and what are the rates of change in these directions?*

Solution

(a) The direction of $\mathbf{v}$ is obtained by dividing $\mathbf{v}$ by its length:

$$|\mathbf{v}| = \sqrt{(2)^2 + (-3)^2 + (6)^2} = \sqrt{49} = 7$$
$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}.$$

The partial derivatives of f at P_0 are
$f_x = (3x^2 - y^2)_{(1,1,0)} = 2, \quad f_y = -2xy|_{(1,1,0)} = -2, \quad f_z = -1|_{(1,1,0)} = -1.$
The gradient of f at P_0 is $\nabla f|_{(1,1,0)} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$. The derivative of f at P_0 in the direction of $\mathbf{v}$ is therefore

$$(D_{\mathbf{u}} f)_{(1,1,0)} = \nabla f|_{(1,1,0)} \cdot \mathbf{u} = (2\mathbf{i} - 2\mathbf{j} - \mathbf{k}) \cdot \left(\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}\right)$$
$$= \frac{4}{7} + \frac{6}{7} - \frac{6}{7} = \frac{4}{7}.$$

(b) The function increases most rapidly in the direction of $\nabla f = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and decreases most rapidly in the direction of $-\nabla f$. The rates of change in the directions are, respectively,

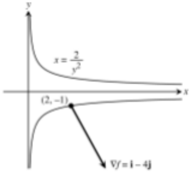
$$|\nabla f| = \sqrt{(2)^2 + (-2)^2 + (-1)^2} = \sqrt{9} = 3 \quad \text{and} \quad -|\nabla f| = -3.$$

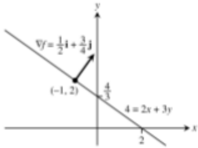
Exercise 13.

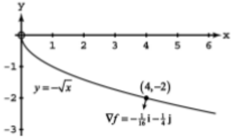
Find the gradient of the function at the given point. Then sketch the gradient together with the level curve that passes through the point.

1. $g(x, y) = xy^2, \quad (2, -1)$
2. $f(x, y) = \sqrt{2x + 3y}, \quad (-1, 2)$
3. $f(x, y) = \tan^{-1} \frac{\sqrt{x}}{y}, \quad (4, -2)$

1. $\frac{\delta g}{\delta x} = y^2 \Rightarrow \frac{\delta g}{\delta x}(2,-1) = 1; \frac{\delta g}{\delta y} = 2x\ y \Rightarrow \frac{\delta g}{\delta x}(2,-1) = -4; \Rightarrow \nabla g = i - 4j; g(2,-1) = 2 \Rightarrow x = \frac{2}{y^2}$ is the level curve.
2. $\frac{\delta f}{\delta x} = \frac{1}{\sqrt{2x+3y}} \Rightarrow \frac{\delta f}{\delta x}(-1,2) = \frac{1}{2}; \frac{\delta f}{\delta y} = \frac{3}{2\sqrt{2x+3y}} = \frac{\delta f}{\delta x}(-1,2) = \frac{3}{4}; \Rightarrow \nabla f = \frac{1}{2}i + \frac{3}{4}j; f(-1,2) = 2 \Rightarrow 4 = 2x + 3y$ is the level curve.
3. $\frac{\delta f}{\delta x} = \frac{y}{2y^2\sqrt{x}+2x^{3/2}} \Rightarrow \frac{\delta f}{\delta x}(4,-2) = -\frac{1}{16}; \frac{\delta f}{\delta y} = \frac{\sqrt{x}}{2y^2+x} \Rightarrow \frac{\delta f}{\delta y}(4,-2) = -\frac{1}{4} \Rightarrow \nabla f = -\frac{1}{16}i - \frac{1}{4}j; f(4,-2) = -\frac{\pi}{4} \Rightarrow y = -\sqrt{x}$ is the level curve.







Exercise 14.

Find ∇f at the given point.

1. $f(x, y, z) = 2z^3 - 3(x^2 + y^2)z + \tan^{-1} xz, \quad (1, 1, 1)$
2. $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2} + \ln(xyz), \quad (-1, 2, -2)$
3. $f(x, y, z) = e^{x+y} \cos z + (y + 1) \sin^{-1} x, \quad (0, 0, \pi/6)$

Solution for the Exercise 14

1. $\frac{\delta f}{\delta x} = -6xz + \frac{z}{x^2z^2+1} \Rightarrow \frac{\delta f}{\delta x}(1,1,1) = -\frac{11}{2}$; $\frac{\delta f}{\delta y} = -6yz \Rightarrow \frac{\delta f}{\delta y}(1,1,1) = -6$; $\frac{\delta f}{\delta z} = 6z^2 - 3(x^2+y^2) + \frac{x}{x^2z^2+1} \Rightarrow \frac{\delta f}{\delta z}(1,1,1) = \frac{1}{2}$; *thus* $\nabla f = -\frac{11}{2}i - 6j + \frac{1}{2}k$
2. $\frac{\delta f}{\delta x} = -\frac{x}{(x^2+y^2+z^2)^{3/2}} + \frac{1}{x} \Rightarrow \frac{\delta f}{\delta x}(-1,2,-2) = -\frac{26}{27}$; $\frac{\delta f}{\delta y} = -\frac{y}{(x^2+y^2+z^2)^{3/2}} + \frac{1}{y} \Rightarrow \frac{\delta f}{\delta y}(-1,2,-2) = \frac{23}{54}$; $\frac{\delta f}{\delta z} = -\frac{z}{(x^2+y^2+z^2)^{3/2}} + \frac{1}{z} \Rightarrow \frac{\delta f}{\delta z}(-1,2,-2) = \frac{23}{54}$; *thus* $\nabla f = -\frac{26}{27}i + \frac{23}{54}j - \frac{23}{54}k$
3. $\frac{\delta f}{\delta x} = e^{x+y}\cos z + \frac{y+1}{\sqrt{1-x^2}} \Rightarrow \frac{\delta f}{\delta x}(0,0,\frac{\pi}{6}) = \frac{\sqrt{3}}{2} + 1$; $\frac{\delta f}{\delta y} = e^{x+y}\cos z + \sin^{-1} x \Rightarrow \frac{\delta f}{\delta y}(0,0,\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$; $\frac{\delta f}{\delta z} = -e^{x+y}\sin z \Rightarrow \frac{\delta f}{\delta z}(0,0,\frac{\pi}{6}) = -\frac{1}{2}$; *thus* $\nabla f = (\frac{\sqrt{3}+2}{2})i + \frac{\sqrt{3}}{2}j - \frac{1}{2}k$

Exercise 15.

1. *What is the derivative of a function $f(x, y)$ at a point P_0 in the direction of a unit vector* u *? What rate does it describe? What geometric interpretation does it have? Give examples.*
2. *What is the gradient vector of a differentiable function $f(x, y)$? How is it related to the function's directional derivatives? State the analogous results for functions of three independent variables.*
3. *How do you find the tangent line at a point on a level curve of a differentiable function $f(x, y)$? Give an example.*
4. *Find the derivative of the function at P_0 in the direction of* u.
   (a) $f(x, y) = 2xy - 3y^2$, $P_0(5, 5)$, $\mathrm{u} = 4\mathrm{i} + 3\mathrm{j}$
   (b) $h(x, y) = \tan^{-1}(y/x) + \sqrt{3}\sin^{-1}(xy/2)$, $P_0(1, 1)$, $\mathrm{u} = 3\mathrm{i} - 2\mathrm{j}$
   (c) $f(x, y, z) = xy + yz + zx$, $P_0(1, -1, 2)$, $\mathrm{u} = 3\mathrm{i} + 6\mathrm{j} - 2\mathrm{k}$
   (d) $g(x, y, z) = 3e^x \cos yz$, $P_0(0, 0, 0)$, $\mathrm{u} = 2\mathrm{i} + \mathrm{j} - 2\mathrm{k}$

Solution for (4.) in Exercise 15

(a) $u = \frac{A}{|A|} = \frac{4i+3j}{\sqrt{4^2+3^2}} = \frac{4}{5}i + \frac{3}{5}j$; $f_x(x,y) = 2y \Rightarrow f_x(5,5) = 10$; $f_y(x,y) = 2x - 6y \Rightarrow$ $f_y(5,5) = -20 \Rightarrow \nabla f = 10i - 20j \Rightarrow (D_u f)_{pu} = \nabla f.u = 10(\frac{4}{5}) - 20(\frac{3}{5}) = -4$

(b) $u = \frac{A}{|A|} = \frac{3i-2j}{\sqrt{3^2+(-2)^2}} = \frac{3}{\sqrt{13}}i - \frac{2}{\sqrt{13}}j$; $h_x(x,y) = \frac{(x^2)}{(\frac{y}{x})^2+1} + \frac{(\frac{y}{x})\sqrt{3}}{\sqrt{1-(\frac{x^2y^2}{4})}} \Rightarrow h_x(1,1) =$ $\frac{1}{2}$; $h_y(x,y) = \frac{(\frac{1}{x})}{(\frac{y}{x})^2+1} + \frac{(\frac{x}{2})\sqrt{3}}{\sqrt{1-(\frac{x^2y^2}{4})}} = h_y(1,1) = \frac{3}{2} \Rightarrow \nabla h = \frac{1}{2}i + \frac{3}{2}j \Rightarrow (D_u h)_{P_D} = \nabla h.u =$ $\frac{3}{2\sqrt{13}} - \frac{6}{2\sqrt{13}} = -\frac{3}{2\sqrt{13}}$

(c) $u = \frac{A}{|A|} = \frac{3i+6j-2k}{\sqrt{3^2+6^2+(-2)^2}} = \frac{3}{7}i + \frac{6}{7}j - \frac{2}{7}k$; $f_x(x,y,z) = y + z \Rightarrow f_x(1,-1,2) =$ 1; $f_y(x,y,z) = x + z \Rightarrow f_y(1,-1,2) = 3$; $f_z(x,y,z) = y + x \Rightarrow f_z(1,-1,2) = 0 \Rightarrow \nabla f =$ $i + 3j \Rightarrow (D_u f)_{P_0} = \nabla f.u = \frac{3}{7} + \frac{18}{7} = 3$

(d) $u = \frac{A}{|A|} = \frac{2i+j-2k}{\sqrt{2^2+1^2+(-2)^2}} = \frac{2}{3}i + \frac{1}{3}j - \frac{2}{3}k$; $g_x(x,y,z) = 3e^x \cos yz \Rightarrow g_x(0,0,0) =$ 3; $g_y(x,y,z) = -3ze^x sinyz \Rightarrow g_y(0,0,0) = 0$; $g_z(x,y,z) = -3ye^x \sin yz \Rightarrow g_z(0,0,0) =$ $0 \Rightarrow \nabla g = 3i \Rightarrow (D_u g)_{P_0} = \nabla g.u = 2$

Exercise 16.

Find the directions in which the functions increase and decrease most rapidly at P_0. Then find the derivatives of the functions in these directions.

(a) $f(x,y) = x^2 + xy + y^2,\ P_0(-1,1)$

(b) $f(x,y,z) = \ln xy + \ln yz + \ln xz,\ P_0(1,1,1)$

Solution for the Exercise 16

(a) $\nabla f = (2x+y)i + (x+2y)j \Rightarrow \nabla f(-1,1) = -i + j \Rightarrow u = \frac{\nabla f}{|\nabla f|} = \frac{-i+j}{\sqrt{(-1)2+1^2}} = -\frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j$; f increases most rapidly in the direction $u = -\frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j$ and decreases most rapidly in the direction $-u = \frac{1}{\sqrt{2}}i - \frac{1}{\sqrt{2}}j$; $(D_u f)_{P_0} = \nabla f.u = |\nabla f| = \sqrt{2}$ *and* $(D_{-u} f)_{p_0} = -\sqrt{2}$.

(b) $\nabla f = (\frac{1}{x} + \frac{1}{x})i + (\frac{1}{y} + \frac{1}{y})j + (\frac{1}{z} + \frac{1}{z})k \Rightarrow \nabla f(1,1,1) = 2i + 2j + 2k \Rightarrow u = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{3}}i + \frac{1}{\sqrt{3}}j + \frac{1}{\sqrt{3}}k$; f increases most rapidly in the direction $u = \frac{1}{\sqrt{3}}i + \frac{1}{\sqrt{3}}j + \frac{1}{\sqrt{3}}k$ and decreases most rapidly in the direction $-u = -\frac{1}{\sqrt{3}}i - \frac{1}{\sqrt{3}}j - \frac{1}{\sqrt{3}}k$; $(D_u f)_{p_0} = \nabla f.u = |\nabla f| = 2\sqrt{3}$ *and* $(D_{-u} f)_{p_0} = -2\sqrt{3}$.

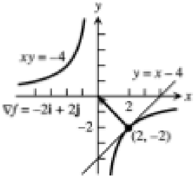
Exercise 17.

Sketch the curve $f(x, y) = c$ together with ∇f and the tangent line at the given point. Then write an equation for the tangent line.

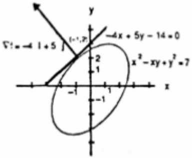
1. $xy = -4, \quad (2, -2)$
2. $x^2 - xy + y^2 = 7, \quad (-1, 2)$

Solution for the Exercise 17

1. $\nabla f = yi + xj \Rightarrow \nabla f(2,-2) = -2i + 2j \Rightarrow$ Tangent line : $-2(x-2) + 2(y+2) = 0 \Rightarrow y = x - 4.$



2. $\nabla f = (2x-y)i + (2y-x)j \Rightarrow \nabla f(-1,2) = -4i + 5j \Rightarrow$ Tangent line : $-4(x+1) + 5(y-2) = 0 \Rightarrow -4x + 5y - 14 = 0.$



Exercise 18.

1. *Let* $f(x, y) = x^2 - xy + y^2 - y$. *Find the directions* u *and the values of* $D_{\mathrm{u}}f(1, -1)$ *for which*
   (a) $D_{\mathrm{u}}f(1, -1)$ *is largest*
   (b) $D_{\mathrm{u}}f(1, -1)$ *is smallest*
   (c) $D_{\mathrm{u}}f(1, -1) = 0$
   (d) $D_{\mathrm{u}}f(1, -1) = 4$
   (e) $D_{\mathrm{u}}f(1, -1) = -3$.
2. *Let* $f(x, y) = \frac{(x-y)}{(x+y)}$. *Find the directions* u *and the values of* $D_{\mathrm{u}}f(-\frac{1}{2}, \frac{3}{2})$ *for which*
   (a) $D_{\mathrm{u}}f(-\frac{1}{2}, \frac{3}{2})$ *is largest*
   (b) $D_{\mathrm{u}}f(-\frac{1}{2}, \frac{3}{2})$ *is smallest*
   (c) $D_{\mathrm{u}}f(-\frac{1}{2}, \frac{3}{2}) = 0$
   (d) $D_{\mathrm{u}}f(-\frac{1}{2}, \frac{3}{2}) = -2$
   (e) $D_{\mathrm{u}}f(-\frac{1}{2}, \frac{3}{2}) = 1$.

Solution for (1.) in Exercise 18

$\nabla f = (2x - y)i + (-x + 2y - 1)j$

(a) $\nabla f(1,-1) = 3i - 4j \Rightarrow |\nabla f(1,-1)| = 5 \Rightarrow D_u f(1,-1) = 5$in the direction of $u = \frac{3}{5}i - \frac{4}{5}j$

(b) $-\nabla f(1,-1) = -3i + 4j \Rightarrow |\nabla f(1,-1)| = 5 \Rightarrow D_u f(1,-1) = -5$ in the direction of $u = -\frac{3}{5}i + \frac{4}{5}j$

(c) $D_u f(1,-1) = 0$ in the direction of $u = \frac{4}{5}i + \frac{3}{5}j$ *or* $u = -\frac{4}{5}i - \frac{3}{5}j$

(d) Let $u = u_1 i + u_2 j \Rightarrow |u| = \sqrt{u_1^2 + u_2^2} = 1 \Rightarrow u_1^2 + u_2^2 = 1$; $D_u f(1,-1) = \nabla f(1,-1) \cdot u = (3i - 4j) \cdot (u_1 i + u_2 j) = 3u_1 - 4u_2 = 4 \Rightarrow u_2 = \frac{3}{4}u_1 - 1 \Rightarrow u_1^2 + (\frac{3}{4}u_1 - 1)^2 = 1 \Rightarrow \frac{25}{16}u_1^2 - \frac{3}{2}u_1 = 0 \Rightarrow u_1 = 0$ *or* $u_1 = \frac{24}{25}$; $u_1 = 0 \Rightarrow u_2 = -1 \Rightarrow u = -j$, or $u_1 = \frac{24}{25} \Rightarrow u_2 = -\frac{7}{25} \Rightarrow u = \frac{24}{25}i - \frac{7}{25}j$

(e) Let $u = u_1 i + u_2 j \Rightarrow |u| = \sqrt{u_1^2 + u_2^2} = 1 \Rightarrow u_1^2 + u_2^2 = 1$; $D_u f(1,-1) = \nabla f(1,-1) \cdot u = (3i - 4j) \cdot (u_1 i + u_2 j) = 3u_1 - 4u_2 = -3 \Rightarrow u_1 = \frac{4}{3}u_2 - 1 \Rightarrow (\frac{4}{3}u_2 - 1)^2 + u_2^2 = 1 \Rightarrow \frac{25}{9}u_2^2 - \frac{8}{3}u_2 = 0 \Rightarrow u_2 = 0$ *or* $u_2 = \frac{24}{25}$; $u_2 = 0 \Rightarrow u_1 = -1 \Rightarrow u = -i$, *or* $u_2 = \frac{24}{25} \Rightarrow u_2 = \frac{7}{25} \Rightarrow u = \frac{7}{25}i + \frac{24}{25}j$

Solution for (2.) in Exercise 18

$\nabla f = \frac{2y}{(x+y)^2} i - \frac{2x}{(x+y)^2} j$

(a) $\nabla f(-\frac{1}{2}, \frac{3}{2}) = 3i + j \Rightarrow |\nabla f(-\frac{1}{2}, \frac{3}{2})| = \sqrt{10} \Rightarrow D_u f(-\frac{1}{2}, \frac{3}{2}) = \sqrt{10}$ in the direction of $u = \frac{3}{\sqrt{10}} i + \frac{1}{\sqrt{10}} j$

(b) $-\nabla f(-\frac{1}{2}, \frac{3}{2}) = -3i - j \Rightarrow |\nabla f(-\frac{1}{2}, \frac{3}{2})| = \sqrt{10} \Rightarrow D_u f(1, -1) = -\sqrt{10}$ in the direction of $u = -\frac{3}{\sqrt{10}} i - \frac{1}{\sqrt{10}} j$

(b) $D_u f(-\frac{1}{2}, \frac{3}{2}) = 0$ in the direction of $u = \frac{1}{\sqrt{10}} i - \frac{3}{\sqrt{10}} j$ *or* $u = -\frac{1}{\sqrt{10}} i + \frac{3}{\sqrt{10}} j$

(c) Let $u = u_1 i + u_2 j \Rightarrow |u| = \sqrt{u_1^2 + u_2^2} = 1 \Rightarrow u_1^2 + u_2^2 = 1; D_u f(-\frac{1}{2}, \frac{3}{2}) = \nabla f(-\frac{1}{2}, \frac{3}{2}) \cdot u = (3i + j) \cdot (u_1 i + u_2 j) = 3u_1 + u_2 = -2 \Rightarrow u_2 = -3u_1 - 2 \Rightarrow u_1^2 + (-3u_1 - 2)^2 = 1 \Rightarrow 10u_1^2 + 12u_1 + 3 = 0 \Rightarrow u_1 = \frac{-6 \pm \sqrt{6}}{10} u_1 = \frac{-6+\sqrt{6}}{10} \Rightarrow u_2 = \frac{-2-3\sqrt{6}}{10} \Rightarrow u = \frac{-6+\sqrt{6}}{10} i + \frac{-2-3\sqrt{6}}{10} j$, *or* $u_1 = \frac{-6-\sqrt{6}}{10} \Rightarrow u_2 = \frac{-2+3\sqrt{6}}{10} \Rightarrow u = \frac{-6-\sqrt{6}}{10} i + \frac{-2+3\sqrt{6}}{10} j$

(d) Let $u = u_1 i + u_2 j \Rightarrow |u| = \sqrt{u_1^2 + u_2^2} = 1 \Rightarrow u_1^2 + u_2^2 = 1; D_u f(-\frac{1}{2}, \frac{3}{2}) = \nabla f(-\frac{1}{2}, \frac{3}{2}) \cdot u = (3i + j) \cdot (u_1 i + u_2 j) = 3u_1 + u_2 = 1 \Rightarrow u_2 = 1 - 3u_1 \Rightarrow u_1^2 + (1 - 3u_1)^2 = 1 \Rightarrow 10u_1^2 - 6u_1 = 0 \Rightarrow u_1 = 0$ *or* $u_1 = \frac{3}{5}; u_1 = 0 \Rightarrow u_2 = 1 \Rightarrow u = j$, *or* $u_1 = \frac{3}{5} \Rightarrow u_2 = -\frac{4}{5} \Rightarrow u = \frac{3}{5} i - \frac{4}{5} j$

Exercise 19.

1. Zero directional derivative : *In what direction is the derivative of* $f(x, y) = xy + y^2$ *at* $P(3, 2)$ *equal to zero?*
2. Zero directional derivative : *Is there a direction* u *in which the rate of change of* $f(x, y) = x^2 - 3xy + 4y^2$ *at* $P(1, 2)$ *equals* 14*? Give reasons for your answer.*

Solution for the Exercise 19

1. $\nabla f = yi + (x+2y)j \Rightarrow \nabla f(3,2) = 2i + 7j$; a vector orthogonal to ∇f *is* $v = 7i - 2j \Rightarrow u = \frac{v}{|v|} = \frac{7i-2j}{\sqrt{7^2+(-2)^2}} = \frac{7}{\sqrt{53}}i - \frac{2}{\sqrt{53}}j$ *and* $-u = -\frac{7}{\sqrt{53}}i + \frac{2}{\sqrt{53}}j$ are the directions where the derivative is zero.
2. $\nabla f = (2x-3y)i + (-3x+8y)j \Rightarrow \nabla f(1,2) = -4i + 13j \Rightarrow |\nabla f(1,2)| = \sqrt{(-4)^2 + (13)^2} = \sqrt{185}$; no, the maximum rate of change is $\sqrt{185} < 14$.

Exercise 20.

1. Changing temperature along a circle : *Is there a direction* u *in which the rate of change of the temperature function* $T(x, y, z) = 2xy - yz$ *(temperature in degrees Celsius, distance in feet) at* $P(1, -1, 1)$ *is* -3 *deg.Cel/ft ? Give reasons for your answer.*
2. *The derivative of* $f(x, y, z)$ *at a point* P *is greatest in the direction of* $\mathrm{v} = \mathrm{i} + \mathrm{j} - \mathrm{k}$. *In this direction, the value of the derivative is* $2\sqrt{3}$.
   (a) *What is* ∇f *at* P *? Give reasons for your answer.*
   (b) *What is the derivative of* f *at* P *in the direction of* $\mathrm{i} + \mathrm{j}$ *?*

Solution for the Exercise 20

1. $\nabla T = 2yi + (2x - z)j - yk \Rightarrow \nabla T(1, -1, 1) = -2i + j + k \Rightarrow |\nabla T(1, -1, 1)| = \sqrt{(-2)^2 + 1^2 + 1^2} = \sqrt{6}$; no, the minimum rate of change is $-\sqrt{6} > -3$
2. (a) $(D_u f)_p = 2\sqrt{3} \Rightarrow |\nabla f| = 2\sqrt{3}; u\frac{v}{|v|} = \frac{i+j-k}{\sqrt{1^2+1^2+(-1)^2}} = \frac{1}{\sqrt{3}}i + \frac{1}{\sqrt{3}}j - \frac{1}{\sqrt{3}}k;$ *thus* $u = \frac{\nabla f}{|\nabla f|} \Rightarrow \nabla f = |\nabla f|u \Rightarrow \nabla f = 2\sqrt{3}(\frac{1}{\sqrt{3}}i + \frac{1}{\sqrt{3}}j - \frac{1}{\sqrt{3}}k) = 2i + 2j - 2k$

   (b) $v = i + j \Rightarrow u = \frac{v}{|v|} = \frac{i+j}{\sqrt{1^2+1^2}} = \frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j \Rightarrow (D_u f)_{P_0} = \nabla f.u = 2(\frac{1}{\sqrt{2}}) + 2(\frac{1}{\sqrt{2}}) = 2(0) = 2\sqrt{2}$

Exercise 21.

1. Directional derivatives and scalar components : *How is the derivative of a differentiable function* $f(x, y, z)$ *at a point* P_0 *in the direction of a unit vector* u *related to the scalar component of* $(\nabla f)_{P_0}$ *in the direction of* u*? Give reasons for your answer.*
2. Directional derivatives and partial derivatives : *Assuming that the necessary derivatives of* $f(x, y, z)$ *are defined, how are* $D_{\mathrm{i}}f$, $D_{\mathrm{j}}f$, *and* $D_{\mathrm{k}}f$ *related to* f_x, f_y, *and* f_z*? Give reasons for your answer.*

Solution for the Exercise 21

1. The directional derivative is the scalar component. With ∇f in the direction of u is $\nabla f . u = (D_u f)_{p_0}$.
2. $D_i f = \nabla f . i = (f_x i + f_y j + f_z k) \cdot i = f_x$; similarly, $D_j f = \nabla f \cdot j = f_y$ and $D_k f = \nabla f \cdot k = f_z$

Exercise 22.

1. Lines in the xy-plane : *Show that* $A(x - x_0) + B(y - y_0) = 0$ *is an equation for the line in the xy-plane through the point* (x_0, y_0) *normal to the vector* $\mathrm{N} = A\mathrm{i} + B\mathrm{j}$.
2. The algebra rules for gradients : *Given a constant* k *and the gradients*

$$\nabla f = \frac{\partial f}{\partial x}\mathrm{i} + \frac{\partial f}{\partial y}\mathrm{j} + \frac{\partial f}{\partial z}\mathrm{k}, \quad \nabla g = \frac{\partial g}{\partial x}\mathrm{i} + \frac{\partial g}{\partial y}\mathrm{j} + \frac{\partial g}{\partial z}\mathrm{k},$$

establish the algebra rules for gradients.

Proof for (1.) in Exercise 22

1. If (x, y) is a point on the line, then $T(x, y) = (x - x_0)i + (y - y_0)j$ is a vector parallel to the line
   $\Rightarrow T \cdot N = 0 \Rightarrow A(x - x_0) + B(y - y_0) = 0$, as claimed.

Proof for (2.) in Exercise 22

(a) $\nabla(kf) = \frac{\delta(kf)}{\delta x} i + \frac{\delta(kf)}{\delta y} j + \frac{\delta(kf)}{\delta z} k = k(\frac{\delta f}{\delta x})i + k(\frac{\delta f}{\delta y})j + k(\frac{\delta f}{\delta z})k = k(\frac{\delta f}{\delta x} i + \frac{\delta f}{\delta y} j + \frac{\delta f}{\delta z} k) = k\nabla f$

(b) $\nabla(f+g) = \frac{\delta(f+g)}{\delta x} i + \frac{\delta(f+g)}{\delta y} j + \frac{\delta(f+g)}{\delta z} k = (\frac{\delta f}{\delta x} + \frac{\delta g}{\delta x})i + (\frac{\delta f}{\delta y} + \frac{\delta g}{\delta y})j + (\frac{\delta f}{\delta z} + \frac{\delta g}{\delta z})k =$
$\frac{\delta f}{\delta x} i + \frac{\delta g}{\delta x} i + \frac{\delta f}{\delta y} j + \frac{\delta g}{\delta y} j + \frac{\delta f}{\delta z} k + \frac{\delta g}{\delta z} k = (\frac{\delta f}{\delta x} i + \frac{\delta f}{\delta y} j + \frac{\delta f}{\delta z} k) + (\frac{\delta g}{\delta x} i + \frac{\delta g}{\delta y} j + \frac{\delta g}{\delta z} k) = \nabla f + \nabla g$

(c) $\nabla(f-g) = \nabla f - \nabla g$(Substite $-g$ for in part (b) above)

(d) $\nabla(fg) = \frac{\delta(fg)}{\delta x} i + \frac{\delta(fg)}{\delta y} j + \frac{\delta(fg)}{\delta z} k = \left(\frac{\delta f}{\delta x} g + \frac{\delta g}{\delta x} f\right) i + \left(\frac{\delta f}{\delta y} g + \frac{\delta g}{\delta y} f\right) j + \left(\frac{\delta f}{\delta z} g + \frac{\delta g}{\delta z} f\right) k =$
$\left(\frac{\delta f}{\delta x} g\right) i + \left(\frac{\delta g}{\delta x} f\right) i + \left(\frac{\delta f}{\delta y} g\right) j + \left(\frac{\delta g}{\delta y} f\right) j + \left(\frac{\delta f}{\delta z} g\right) k + \left(\frac{\delta g}{\delta z} f\right) k =$
$f\left(\frac{\delta g}{\delta x} i + \frac{\delta g}{\delta y} j + \frac{\delta g}{\delta z} k\right) + g\left(\frac{\delta f}{\delta x} i + \frac{\delta f}{\delta y} j + \frac{\delta f}{\delta z} k\right) = f\nabla g + g\nabla f$

(e) $\nabla(\frac{f}{g}) = \frac{\delta(\frac{f}{g})}{\delta x} i + \frac{\delta(\frac{f}{g})}{dy} j + \frac{\delta(\frac{f}{g})}{\delta z} k = \left(\frac{g\frac{\delta f}{\delta x} - f\frac{\delta g}{\delta x}}{g^2}\right) i + \left(\frac{g\frac{\delta f}{\delta y} - f\frac{\delta g}{\delta y}}{g^2}\right) j + \left(\frac{g\frac{\delta f}{\delta z} - f\frac{\delta g}{\delta z}}{g^2}\right) k =$
$\left(\frac{g\frac{\delta f}{\delta x} i + g\frac{\delta f}{\delta y} j + g\frac{\delta f}{\delta z} k}{g^2}\right) - \left(\frac{f\frac{\delta g}{\delta x} i + f\frac{\delta g}{\delta y} j + f\frac{\delta g}{\delta z} k}{g^2}\right) =$
$g\left(\frac{\frac{\delta f}{\delta x} i + g\frac{\delta f}{\delta y} j + g\frac{\delta f}{\delta z} k}{g^2}\right) - f\left(\frac{\frac{\delta g}{\delta x} i + f\frac{\delta g}{\delta y} j + f\frac{\delta g}{\delta z} k}{g^2}\right) = \frac{g\nabla f}{g^2} - \frac{f\nabla g}{g^2} = \frac{g\nabla f - f\nabla g}{g^2}.$

Exercise 23.

1. *Find the directions in which the functions increase and decrease most rapidly at P_0. Then find the derivatives of the functions in these directions. Also find the dertivative of f at P_0 in the direction of the vector* v.
   (a) $f(x, y) = x^2e^{-2y}$, $P_0(1, 0)$, $\mathrm{v} = \mathrm{i} + \mathrm{j}$
   (b) $f(x, y, z) = \ln(2x + 3y + 6z)$, $P_0(-1, -1, 1)$, $\mathrm{v} = 2\mathrm{i} + 3\mathrm{j} + 6\mathrm{k}$
2. *What is the largest value that the directional derivative of $f(x, y, z) = xyz$ can have at the point $(1, 1, 1)$?*

Exercise 24.

1. *The temperature of a point in space is given by* $T(x,y,z) = x^2 + y^2 - z$. *A mosquito located at* $(1,1,2)$ *desires to fly in such a direction that it will get warm as soon as possible. In what direction should it move?*
2. *Prove the following :*
   (a) $\nabla r^n = nr^{n-2}\mathrm{r}$
   (b) $\nabla \frac{1}{r} = \frac{-\mathrm{r}}{r^3}$
   (c) $\nabla \ln|\mathrm{r}| = \frac{\mathrm{r}}{r^2}$ *where* $\mathrm{r} = x\mathrm{i} + y\mathrm{j} + z\mathrm{k}$, $|\mathrm{r}| = r$ *and* n *is an integer.*
3. *Find the directional derivative of* $f(x,y,z) = x^2 - y^2 + 2z^2$ *at the point* $P(1,2,3)$ *in the direction of the line* PQ *where* Q *is the point* $(5,0,4)$. *Also, calculate the magnitude of the maximum directional derivative.*

Weather Map

The weather map in the Figure shows a contour map of the temperature function $T(x, y)$ for the states of California and Nevada at 3 : 00 PM on a day in October. The level curves, or isothermals, join locations with the same temperature.
The partial derivative T_x at a location such as Reno is the rate of change of temperature with respect to distance if we travel east from Reno; T_y is the rate of change of temperature if we travel north. But what if we want to know the rate of change of temperature when we travel southeast (toward Las Vegas), or in some other direction?

Directional derivative enables us to find the rate of change of a function of two or more variables in any direction.

Weather Map

Exercise 25.

Use the weather map in the above figure to estimate the value of the directional derivative of the temperature function at Reno in the southeasterly direction.

Solution :

The unit vector directed toward the southeast is $u = (\mathrm{i} - \mathrm{j})/\sqrt{2}$, but we won't need to use this expression.

We start by drawing a line through Reno toward the south-east.

Solution (contd...)

We approximate the directional derivative $D_u T$ by the average rate of change of the temperature between the points where this line intersects the isothermals $T = 50$ and $T = 60$.

The temperature at the point southeast of Reno is $T = 60^\circ F$ and the temperature at the point northwest of Reno is $T = 50^\circ F$.

The distance between these points looks to be about 75 miles. So the rate of change of the temperature in the southeasterly direction is

$$D_u T \approx \frac{60 - 50}{75} = \frac{10}{75} \approx 0.13^\circ F/mi.$$

Exercise 26.

The directional derivative of the function $f(x, y, z) = 3xy + z^2$ at the point $P_0(1, -2, 2)$ in the direction from the point P_0 towards the origin is

(a) $\frac{4}{3}$

(b) $\frac{3}{4}$

(c) $-\frac{3}{4}$

(d) $-\frac{4}{3}$

Correct Answer : $\frac{4}{3}$

Old Questions

Which of the following statements is/are false?

Select one or more:

- ☐ A.
  If f has zero change at (x_0, y_0) in the direction of $\mathbf{u}$, then $\mathbf{u}$ is orthogonal to $(\nabla f)_{(x_0,y_0)}$.
- ☐ B.
  If f is differentiable at (x_0, y_0), then derivative of f exists at (x_0, y_0) in any direction.
- ☐ C.
  The directional derivative of f at (x_0, y_0) in the direction of unit vector $\mathbf{u}$ is $(\nabla f)_{(x_0,y_0)} \cdot \mathbf{u}$.
- ☐ D.
  If $(\nabla f)_{(x_0,y_0)}$ is the zero vector, then f is not differentiable at (x_0, y_0).

Your answer is incorrect.

The correct answers are:
If $(\nabla f)_{(x_0,y_0)}$ is the zero vector, then f is not differentiable at (x_0, y_0).

,

The directional derivative of f at (x_0, y_0) in the direction of unit vector $\mathbf{u}$ is $(\nabla f)_{(x_0,y_0)} \cdot \mathbf{u}$.

,

If f has zero change at (x_0, y_0) in the direction of $\mathbf{u}$, then $\mathbf{u}$ is orthogonal to $(\nabla f)_{(x_0,y_0)}$.

Q.2. At the point $(1, 2)$, if a function $f(x, y)$ has a derivative of 2 in the direction towards $(2, 2)$ and a derivative of -2 in the direction towards $(1, 1)$, then the direction(s) in which f changes rapidly at $(1, 2)$ is/are

(a) $\frac{\hat{\boldsymbol{i}}}{\sqrt{2}} - \frac{\hat{\boldsymbol{j}}}{\sqrt{2}}$ (b) $\frac{\hat{\boldsymbol{i}}}{\sqrt{2}} + \frac{\hat{\boldsymbol{j}}}{\sqrt{2}}$ (c) $-\frac{\hat{\boldsymbol{i}}}{\sqrt{2}} + \frac{\hat{\boldsymbol{j}}}{\sqrt{2}}$ (d) $-\frac{\hat{\boldsymbol{i}}}{\sqrt{2}} - \frac{\hat{\boldsymbol{j}}}{\sqrt{2}}$

Solution: (b) and (d)

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